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Coupling of branes and normalization of effective actions in string/M-theory

S. P. de Alwis[1]

Department of Physics, Box 390, University of Colorado, Boulder, CO 80309

Abstract

We discuss issues involving p (D)- brane charge quantization and the normalization of effective actions, in string/M-theory. We also construct the action of (the bosonic sector of) eleven dimensional supergravity in the presence of two and five branes and discuss (perturbative) anomaly cancellation.

[1]e-mail: dealwis@gopika.colorado.edu

1 Introduction

A major claim of string theory is that it has no parameters. All of the fundamental experimentally measureable constants should in principle be determined by the dynamics. This means in particular that the string coupling, which is the expectation value of the scalar dilaton field, as well as the size and shape of the six-dimensional compact space, should be fixed by some (hitherto unknown) mechanism. Clearly these values are not determined in perturbation theory and it is therefore important to understand the non-perturbative dynamics of strings. The discovery of various strong-weak coupling relations (S-dualities) in string theory, as well as the connection of string theory to M-theory [1], leads one to hope that one may be on the threshold of understanding the non-perturbative dynamics of string theory. Coupled with the insight coming from the incorporation of D-branes[2] into string theory[1] it seems that one can fix (non-perturbatively) the ten dimensional gauge and gravitational constants in terms of the dilaton expectation value and the string scale.

Some work on these issues was done by this author in reference [5]. Part of the motivation for this paper is to elaborate on those arguments, particularly in relation to the claim that these constants are fixed non-perturbatively at the string scale. The idea is that these constants are the exact values that go into the low energy string effective action (obtained by integrating out the sub-string scale fluctuations) which is used as a starting point for compactification and renormalization group evolution down to low energies. The paper also contains a detailed discussion of the coupling of two and five-branes to background supergravity (with two-branes that may end on five-branes) in an explicit manner. In section two (which borrows heavily from [6]) we discuss solutions of string effective actions with p-form gauge fields. The object of the discussion is to show that purely from the demand for the existence of solutions of these actions, and a natural identification of the metric on the brane, one gets the dilaton dependence for the action

[1]For reviews of D-branes with extensive references to the original litereature see [3] and for M-theory see [4].

for the zero modes that is expected for D-branes from lowest order perturbation theory calculations [3] (when the gauge fields are of the R-R type). One also gets the canonical form [1] of the R-R action. As a consequence we argue that the dilaton dependence coming from lowest order perturbation theory is probably valid non-perturbatively. In section three, much of which is a review of earlier work, we use the Dirac quantization relations for p-branes, T-duality, and type IIB $SL(2,Z)$, to show how the tensions of all D-branes, and also of the two and five branes of M-theory, can be expressed in terms of the fundamental length scale. In addition we show that the ten-dimensional gravitational coupling of string theory and the eleven dimensional gravitational coupling of M-theory, are determined in terms of the string scale. We subject these relations to various consistency checks. Again the main point is that these relations are exactly (non-perturbatively) valid at the string scale. While these relations have been presented before [2],[5] the detailed discussion given here will hopefully further clarify the issues involved. In section four we present arguments to the effect that the form of the gauge field dependence in the D-brane action is also valid non-perturbatively. In section five we extend our discussion to theories which may be phenomenologically relevant. In section six we discuss the coupling of two- and five-branes in M-theory (with two-branes which may sit on five-branes [8],[9]) to the low energy M-theory background. In section seven we use this formalism to discuss (perturbative) anomaly cancellation in the presence of five-branes and in particular suggest a possible resolution of a puzzle pointed out recently [7].

2 Solutions of effective actions

The bosonic part of the closed superstring action is given by

$$S = -T\int_{W_2} d^2\xi \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^M \partial_j X^N G_{MN} - T\int_{W_2} P(B_2) - 1\omega ver 4\pi \int_{W_2} \phi R^{(2)}. \quad (2.1)$$

The tension $T = \frac{1}{2\pi\alpha'}$, where $l_s \equiv \sqrt{\alpha'}$ is the associated string (length) scale, is a physical quantity which is measurable in principle on the assumption that strings can exist as

isolated objects i.e. like electrons and unlike quarks.[2] The above action describes a string (whose world sheet is W_2) in a background consisting of a condensate of its massless bosonic fluctuations, giving the metric (G) the NS-NS two form field (B) [3]and the dilaton scalar field (ϕ). The string coupling is given by

$$g =< e^{\phi} >_0 . \tag{2.2}$$

The coupling of the theory is not a free parameter but is determined dynamically. In perturbative string theory this value is actually undetermined and one has to appeal to non-perturbative effects, which are certainly far from being understood at this point, in order to fix it. Later on we will give two different identifications of this value, which are consistent with each other and T-duality, in terms of ratios of physical lengths.

The quantum consistency of this world sheet action imposes the requirement that

$$T \int_{C_2} H_3 = 2\pi n, \quad n \epsilon \mathcal{Z}, \tag{2.3}$$

where locally $H_3 = dB$, and this fixes the normalization of B in terms of the string tension. Also as long as the world sheet is closed (no open-closed string interactions) the action (2.1) is invariant under the guage transformation $B \rightarrow B + d\Lambda$. Now unlike a particle, a string can only propagate in backgrounds which satisfy the requirement of conformal invariance, and this leads in a well-known fashion to β-function equations that are effective equations of motion for the background fields. These equations imply the existence of an effective action for the condensed massless fields that takes the form (keeping only the leading terms in an α' expansion)

$$I = -\frac{1}{2\kappa^2} \int_{M_{10}} \sqrt{-G} e^{-2\phi} \left\{ R - 4(\nabla\phi)^2 + \frac{1}{12} H^2 \right\} . \tag{2.4}$$

[2]This is assumed to be the case even though the string coupling is probably fixed at some value of order one at the string scale (rather than 10^{-2} as in QED).

[3] $P(B) = \frac{1}{2!} B_{MN} \frac{\partial X^M}{\partial \xi^i} \frac{X^N}{\partial \xi^j} d\xi^I d\xi^j$ is the pull back of the target space field to the world sheet. Our convention for n-forms is $A_n = \frac{1}{n!} A_{i_1,\dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$, $F_{n+1} = dA_n = \frac{1}{n!} \partial_{i_0} A_{i_1,\dots i_n} dx^{i_0} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_n}$.

It is important to note that the relative coefficients in this action are fixed by world sheet conformal invariance. Also, although the Lagrangian density apears with an overall dilaton factor appropriate to a spherical world sheet, since the beta function equations are independent of global topology one expects this action to be valid to all orders in perturbation theory[4], and is perhaps even an exact statement of the interactions of these fields at low energies. For the rest of this paper we will assume that the latter is actually the case.

The target space metric G that occurs in the above formulae is called the string metric since it is the metric in which (upon elimination of the intrinsic metric in (2.1) the string action is given by the area of the world sheet; i.e. as $\int_{W_2}\sqrt{-G}$ [5]. From (2.4) we see that the ten dimensional gravitational constant is given as $G_N^{10}=\frac{\kappa^2 g^2}{8\pi}$. Alternatively one can transform to the so-called Einstein frame in which the effective gravitational action has the standard form. Thus $G^E_{\mu\nu}=e^{-\frac{1}{2}\phi}G_{\mu\nu}$. In this case the gravitational constant is $G_N^{10}=\frac{\kappa^2}{8\pi}$ but the string tension [6] is now $T=\frac{g^{\frac{1}{2}}}{2\pi\alpha'}$. The dimensionless quantity $G_N T^4$ is of course independent of the metric.

An action for a p-brane coupling to a background (p+1)-form field may be written as,

$$S_p=-T_p\int_{W_{p+1}}d^{p+1}\xi e^{c\phi}\sqrt{\det G_{ij}}-T_p\int_{W_{p+1}}A_{p+1}. \tag{2.5}$$

Where G_{ij} is the metric on M_{10} pulled back to the world volume and in the last integral the pullback map to the world volume is understood. c in the above is a constant and the field ϕ need not be identified a'priori with the dilaton introduced earlier. In general it may be a function of all scalar (composite) fields, but in the subsequent analysis the identification with the standard dialaton will be justified and so we will use the same letter here. The only assumption here is that the first term is given by a positive function of

[4]There are terms which may arise due to the so-called Fischler-Susskind effect, coming from degenerate Riemann surfaces, but these are not expected to modify the terms above.

[5]Wherever there is no confusion we will use the same letter for the field defined on M_{10} and the pulled-back field on a p-brane world volume W_{p+1}.

[6]In this paper $2\pi\alpha'$ will always be the inverse string tension in the string frame.

the dilaton times the volume form. It is convenient to introduce a current (p+1)-form whose dual is a closed delta function (9-p) form which restricts an integral over M_{10} to W_{p+1}.

$$* J_{p+1} = \delta_{9-p}(M_{10} \to W_{p+1}), \quad d * J = d\delta = 0 \tag{2.6}$$

Note that the last equation is simply a current conservation condition. Also this is normalized such that $\int_{M_{9-p}} \delta = 1$. If for instance we take the p-brane to be embedded in the sub-space spanned by the first p+1 coordinates then an explicit expression for this is

$$* J_{p+1} = \delta_{9-p} = \int_{W_{p+1}} dX^1(\xi) \wedge ... \wedge dX^{p+1} \delta^{10}(x - X(\xi)) dx^{p+2} \wedge ... \wedge dx^{10}. \tag{2.7}$$

The topological term in the above action may then be written as

$$\int_{W_{p+1}} A_{p+1} = \int_{M_{10}} A_{p+1} \wedge \delta = \int_{M_{10}} A_{p+1} \wedge *J. \tag{2.8}$$

Let us now include a kinetic term $\frac{1}{2\kappa^2} \int e^{-b\phi} \frac{1}{2} F_{p+2} \wedge *F_{p+2}$, where $F = dA_{p+1}$, in the low energy effective action (2.4). We couple the p-brane to the background by replacing the action I by $I + S_p$. The equation of motion for A then gives us the generalized Maxwell equation

$$\frac{1}{2\kappa^2} d(e^{-b\phi} * F) = T_p * J, \tag{2.9}$$

whose integrated form is the electric charge equation (Gauss Law),

$$\frac{1}{2\kappa^2} \int_{\partial M_{9-p}} e^{-b\phi} * F_{p+2} = T_p. \tag{2.10}$$

By a standard argument the quantum consistency of the p-brane action gives,

$$T_p \int_{W_{p+2}} F_{p+2} = 2\pi n, \quad n \epsilon \mathcal{Z}. \tag{2.11}$$

Now we may introduce the dual field strength

$$\tilde{F}_{8-p} = e^{-b\phi} * F_{p+2} \tag{2.12}$$

By demanding the quantum consistency of the coupling $T_{6-p}\int_{W_{7-p}} A_{7-p}$ to a (6-p)-brane, we have the well-known generalization of the Dirac quantization condition [10]

$$2\pi n = T_{6-p}\int_{W_{8-p}} \tilde{F}_{8-p} = 2\kappa^2 T_{6-p} T_p, \tag{2.13}$$

where the last step follows from (2.12) and (2.10).

Let us now look at the low-energy effective action for dilaton-gravity and a (p+2) form field strength in the Einstein metric.

$$I = \frac{1}{2\kappa^2}\int_{M_{10}} d^{10}x\sqrt{-G^E}(R - \frac{1}{2}(\nabla\phi)^2) - \frac{1}{2\kappa^2}\int_{M_{10}} e^{-a\phi}\frac{1}{2}F_{p+2}\wedge *F_{p+2}, \tag{2.14}$$

where the curvature and scalar products are all defined in the Einstein metric. Let us now define the p-brane metric as $G^{(p)}_{MN} = e^{a\phi/(p+1)}G^E_{MN}$. Transforming to this metric we find,

$$I = \frac{1}{2\kappa^2}\int_{M_{10}} d^{10}x\sqrt{-G^{(p)}}e^{\frac{-4a\phi}{p+1}}(R - \frac{1}{2}(\nabla\phi)^2) - \frac{1}{2\kappa^2}\int_{M_{10}} e^{\frac{-4a\phi}{p+1}}\frac{1}{2}F_{p+2}\wedge *F_{p+2}. \tag{2.15}$$

Thus in this metric, the effective low energy action scales under constant shifts of the dilaton, in the same way that the dilaton gravity action with the tree-form field strength term, scales in the string metric. It is natural to define the p-brane action in this metric as[7],

$$S_p = -T_p\int_{W_p} d^{p+1}\xi\sqrt{\det G^{(p)}_{ij}} - T_p\int_{W_p} A_{p+1}. \tag{2.16}$$

Now let us require that there exist asymptotically flat solutions to the equations of motion coming from the action $I + S_p$. Then, as shown in the review by Duff et al [6] (see equation (3.38)) this determines

$$a = \pm(p-3)/2. \tag{2.17}$$

Let us take the lower sign in (2.17). Transforming to the 'string' metric by putting

$$G^{(p)}_{MN} = e^{-\frac{(p-3)\phi}{2(p+1)}}G^E_{MN} = e^{-\frac{p-1}{p+1}\phi}G_{MN}. \tag{2.18}$$

[7]An alternative argument is given in [6].

we have

$$I = \frac{1}{2\kappa^2}\int_{M_{10}} \sqrt{-G}e^{-2\phi}\left\{R + 4(\nabla\phi)^2\right\} - \frac{1}{2\kappa^2}\int_{M_{10}} e^{(p-3)\phi}\frac{1}{2}F_{p+2}\wedge *F_{p+2} \tag{2.19}$$

for the effective background action and

$$S_p = -T_p\int_{W_p} d^{p+1}\xi e^{-\frac{p-1}{2}\phi}\sqrt{\det G_{ij}} - T_p\int_{W_p} A_{p+1} \tag{2.20}$$

For $p = 1$, I becomes the NS-NS sector of the string effective action and S becomes the string action in the string metric. This justifies the name for this metric as well as the identification of ϕ with the standard perturbative dilaton. For $p = 5$ we have the dual form of the NS-NS string effective action and the action S_5 has the characteristic $e^{-2\phi}$ dilaton dependence of a solitonic action.

Let us now choose the upper sign and then transform back to the string metric using

$$G^{(p)}_{MN} = e^{\frac{(p-3)\phi}{2(p+1)}}G^E_{MN} = e^{-\frac{2\phi}{p+1}}G_{MN}. \tag{2.21}$$

The low energy effective action then takes the form,

$$I = \frac{1}{2\kappa^2}\int_{M_{10}} \sqrt{-G}e^{-2\phi}\left\{R + 4(\nabla\phi)^2\right\} - \frac{1}{2\kappa^2}\int_{M_{10}} \frac{1}{2}F_{p+2}\wedge *F_{p+2}. \tag{2.22}$$

This is in the form that one expects for the kinetic energy terms of the Ramond-Ramond fields. Transforming the p-brane action to the string metric we get,

$$S_p = -T_p\int_{W_{p+1}} d^{p+1}\xi e^{-\phi}\sqrt{\det G_{ij}} - T_p\int_{W_{p+1}} A_{p+1}. \tag{2.23}$$

This has exactly the form of the D-brane action obtained from string perturbation theory (albeit with the gauge field on the brane set to zero).

What is remarkable here is that this result has been obtained without any use of string perturbation theory or indeed of the existence of open strings. The point is that the low energy effective action is an quantum exact action and all that is assumed is the existence of the relevant p-brane configurations together with a very natural assumption about their actions. Thus the dilaton dependence of the D-action is probably independent of perturbation theory.

3 T-duality arguments and D-brane tensions

We will first briefly review some arguments in given in [5] and elsewhere[8].

Let us consider asymptotically flat configurations with $G_{MN} \to \eta_{MN}$, $\phi \to \phi_0 = const.$ with $e^{\phi_0} = g$ the string coupling constant. The physical tension (energy per unit spatial volume) is then

$$\tau_p = g^{-1} T_p. \tag{3.1}$$

Note that T_p is in fact the electric charge associated with the brane and so the above is a charge-mass relation that follows from the equality of the two coefficients in the two terms of (2.5), which in turn is a consequence of super (κ)-symmetry. Thus it is in fact a BPS formula [11]. Now consider configurations with one (spatial) isometry. Choosing coordinates such that the Killing vector is $\partial/\partial x^{10}$ and taking that direction to to be a circle of radius R, T-duality is the statement that the transformation

$$R \to R' = \frac{\alpha'}{R}, \quad e^{\phi} \to e^{\phi'} = \frac{\sqrt{\alpha'}}{R} e^{\phi}, \tag{3.2}$$

gives us a theory with the same physics (with in particular the same effective action). The requirement that the D-brane physical mass remains the same and the argument that the p-brane is mapped into a (p$\pm$1) brane under T-duality, then gives the relation

$$T_{p-1} = 2\pi\sqrt{\alpha'} T_p. \tag{3.3}$$

Now SL(2,Z) duality of type IIB string theory tells us that the D-string charge is equal to the fundamental string charge (tension) i.e. $T_1 = T = \frac{1}{2\pi\alpha'}$. Thus we have from (3.3) the formula,

$$T_p = \frac{1}{(2\pi)^p (\sqrt{\alpha'})^{p+1}}. \tag{3.4}$$

Using also the Dirac quantization formula with $n = 1$ (2.13)[9], we get

$$2\kappa^2 = (2\pi)^7 \alpha'^4. \tag{3.5}$$

[8]See the first paper of [3] for a review of the relevant literature.

[9]It should be noted that for RR fields and D-branes, we put $b = 0$ in (2.10) and the curvature and scalar products are taken in the string metric.

Thus the quantization rule just serves to fix the relation between the ten dimensional gravitational constant and the string scale (in the string metric) once the ambiguity arising from the ability to do constant shifts in the dilaton is fixed by our choice of the T-duality rules.

Now let us consider the relationship to M-theory. The bosonic part of the low energy M-theory effective action is given by

$$I_{11} = -\frac{1}{2\kappa_{11}^2}\int_M \sqrt{-G}R - \frac{1}{2\kappa_{11}^2}\frac{1}{2}\int_M K_4 \wedge *K_4 - \frac{1}{2\kappa_{11}^2}\frac{1}{6}\int_M C_3 \wedge K_4 \wedge K_4, \qquad (3.6)$$

where K_4 is a closed 4-form field strength which may be locally written as $K_4 = dC_3$ and the integration is over an 11 dimensional Minkowski signature manifold. Let us consider 11D configurations which have an isometry generated by a Killing vector $k = \partial/\partial y$ with the standard identification

$$y \leftrightarrow y + 2\pi\sqrt{\alpha'}. \qquad (3.7)$$

The Kaluza-Klein ansatz for reducing this to the type IIA low-energy effective action[10] is

$$ds^2 = e^{-\frac{2}{3}\phi_A(x)}G_{\mu\nu}(x)dx^\mu dx^\nu + e^{\frac{4}{3}\phi_A(x)}(dy - C_\mu(x)dx^\mu)^2, \qquad (3.8)$$

and

$$\begin{aligned} C_3 &= \frac{1}{3!}C_{MNP}dx^M \wedge dx^N \wedge dx^P = \frac{1}{3!}A_{\mu\nu\lambda}dx^\mu \wedge dx^\nu \wedge dx^\lambda + \frac{1}{2!}B_{\mu\nu}dx^\mu \wedge dx^\nu \wedge dy \\ &= A_3 + B_2 \wedge dy. \end{aligned} \qquad (3.9)$$

At large 10 D distances we can then identify the radius of the eleventh dimension (parametrized by y) with the (IIA) coupling constant by

$$R_{11} = \sqrt{\alpha'} < e^{\frac{2\phi_A}{3}} >= \sqrt{\alpha'}g_A^{\frac{2}{3}}. \qquad (3.10)$$

[10] see [4], [6] and references therein.

This gives a physical definition to the IIA coupling. By using (3.8) and (3.9) in (3.6) one obtains the IIA action [6],

$$\begin{aligned} I_{IIA} &= -\frac{1}{2\kappa^2}\int_{M_{10}}\sqrt{-G}e^{-2\phi}\left\{R-4(\nabla\phi)^2+\frac{1}{12}H^2\right\} \\ &\quad -\frac{1}{2\kappa^2}\int_{M_{10}}(\frac{1}{2}F_2\wedge *F_2+\frac{1}{2}F_4\wedge *F_4)-\frac{1}{2\kappa^2}\frac{1}{2}\int_{M_{10}}F_4'\wedge F_4'\wedge B_2, \end{aligned} \tag{3.11}$$

where

$$2\kappa_{11}^2=(2\pi\sqrt{\alpha'})2\kappa^2,\ H_3=dB_2,\ F_2=dC_1,\ F_4'=dC_3,\ F_4=F_4'-H_3\wedge C_1. \tag{3.12}$$

The M-theory membrane has for its bosonic part the action

$$S_2=-T_2^M\int_{W_3}d^3\xi\sqrt{\det G_{ij}^M}-T_2^M\int_{W_3}C_3, \tag{3.13}$$

where the appropriate pull-back maps to the world volume are understood. Let us first consider the case of double dimensional reduction where the membrane is wrapped around the eleventh dimension to give a string. Using the pull-back of the Kaluza-Klein ansatze (and choosing W_3 coordinates $\xi^i=(\sigma^1,\sigma^2,\rho)$ with winding number $\partial_\rho y=\nu\epsilon\mathcal{Z}$ $(\partial_\rho X^\mu=0)$, one gets [4] the string action

$$S_1=-\nu 2\pi\sqrt{\alpha'}T_2^M\int_{W_2}d^3\xi\sqrt{\det G_{ij}}-\nu 2\pi\sqrt{\alpha'}T_2^M\int_{W_2}B_2. \tag{3.14}$$

Since this must be the fundamantal string with tension (charge) $1/2\pi\alpha'$ we get

$$T_2^M=\frac{1}{(2\pi)^2\alpha'^{\frac{3}{2}}}. \tag{3.15}$$

On the other hand by simple dimensional reduction one can get the D-membrane in ten dimensional IIA string theory [4]. The gauge field on the D-brane is actually the dual of the one form dy coming from the compactified eleventh coordinate field on the membrane [12], [8],[13]. Specifically, a term

$$-T_2^M(2\pi\alpha')\int_{W_3}F_2\wedge(Y_1-C_1) \tag{3.16}$$

where locally $F_2 = dA_1$, is added. Integrating A_1 gives the condition $d(Y - C) = 0$ which is solved by $Y - C = dy$. This gives the original form of the membrane action on $M_{10} \times S_1$ with the background metric. The normalization is the correct one because the integral cohomolgy classes are defined as

$$\frac{[F]}{2\pi}\epsilon\mathcal{Z}, \qquad \frac{[Y-C]}{2\pi\sqrt{\alpha'}}\epsilon\mathcal{Z} \tag{3.17}$$

The first is the standard normalization of the gauge field and the second is our standard identification of the eleventh coordinate (3.7). Then using (3.15) we see that the additional term is just 2π times an integer. By integrating out the field Y one obtains

$$S_2 = -T_2^M \int_{W_3} d^3\xi e^{-\phi}\sqrt{\det(G_{ij} + \mathcal{F}_{2ij})} - T_2^M \int_{W_3} (A_3 + \mathcal{F}_2 \wedge C_1), \tag{3.18}$$

where $\mathcal{F}_2 \equiv 2\pi\alpha' F_2 - B_2$. This is exactly the D-membrane action[11] so that we have from (3.15)

$$T_2 = T_2^M = \frac{1}{(2\pi)^2 \alpha'^{\frac{3}{2}}}. \tag{3.19}$$

The consistency of the above expression with the formula (3.4) is a reflection of the agreement of our conventions for T-duality and the standard identification of the eleventh coordinate (3.7). Hence we may use our earlier calculation of the ten dimensional gravitational constant (3.5) and (3.12) to get,

$$2\kappa_{11}^2 = (2\pi)^8 \alpha'^{\frac{9}{2}}. \tag{3.20}$$

It is also easy to see, by identifying the M-theory five-brane wrapped around the circular eleventh dimension, with the D-four-brane that, $2\pi\sqrt{\alpha'}T_5^M = T_4 = 1/(2\pi)^4 \alpha'^{\frac{5}{2}}$ so that $T_5^M = 1/(2\pi)^4 \alpha'^3 = T_5$. This provides another check on the above relation for κ_{11} since

$$2\kappa_{11}^2 T_2^M T_5^M = (2\pi)^8 \alpha'^{\frac{9}{2}} \frac{1}{(2\pi)^2 \alpha'^{\frac{3}{2}}} \frac{1}{(2\pi)^2 \alpha'^{\frac{3}{2}}} = 2\pi, \tag{3.21}$$

[11] This agreement [13] is only shown at the classical level, and it does not seem possible to extend for instance the argument made for mapping the type IIB SL(2,Z) dual strings to each other quantum mechanically, to the present case [14].

which is exactly the M-theory Dirac quantization condition. For completeness we note finally that the Kaluza-Klein particle mass quantum which in M-theory units is $\tau_0^{(M)} = 1/R_{11}$ in string units has mass $1/(g_A\sqrt{\alpha'}) = \tau_0$ [1], the mass of a D-0-brane.

Let us now check the consistency of the above with the normalizations of the topological terms in string theory effective actions. In the case of IIA the latter is a result of the proper normalization of the topological term in the M-theory action which was checked in [15]. The term in question is (see (3.11))

$$\frac{1}{2\kappa^2}\frac{1}{2}\int_{M_{10}} F_4' \wedge F_4' \wedge B_2. \tag{3.22}$$

By standard arguments the quantum consistency of the IIA low energy action[12] requires that the integral $\frac{1}{2\kappa^2}\frac{1}{2}\int_M F_4' \wedge F_4' \wedge H_3$ (where M is a closed eleven dimensional manifold[13], be 2π times an integer. From (2.11) we have the magnetic charge equations,

$$\int_{S_4} F_4' = \frac{2\pi}{T_2}, \quad \int_{S_3} H_3 = \frac{2\pi}{T_1}. \tag{3.23}$$

Choosing $M = S_4 \times S_4 \times S_3$ we have

$$\frac{1}{2\kappa^2}\frac{1}{2}\int_M F_4' \wedge F_4' \wedge H_3 = \frac{1}{2\kappa^2}\frac{1}{2}2\left(\frac{2\pi}{T_2}\right)^2\frac{2\pi}{T_1} = \frac{[(2\pi)(2\pi)^2\alpha'^{3/2}]^2(2\pi)(2\pi\alpha')}{(2\pi)^7\alpha'^4} = 2\pi. \tag{3.24}$$

In the first equality above the factor two comes from the two ways of integrating F_4 over the two S_4's, the next equality is obtained by substituting from (3.5) and (3.4), and the last equality shows the consistency of the tension formulae with the normalization of the topological terms.[14]

[12] It might be asked why one demands this, since the action in question is already an effective action. The point is that this is an effective action only to the extent that substring scale fluctuations have been integrated out. However to discuss physics well below the string scale, (presumably after compactification), one needs to integrate down further and one therefore should require the above action to be quantum mechanically consistent.

[13] Strictly speaking we should be talking about Euclidean manifolds and the term should appear with a factor i.

[14] It should be stressed that this is not a check of the question of whether the toplogical term is well-defined. For that one needs to consider a general tweleve manifold, and this is done in [15].

There is a similar topological term in the type IIB effective action. In this case of course there is no direct connection to the M-theory action, but there is an alternate argument[15] which we summarize here. Type IIB has two two-form fields B_2 and C_2 from the NS-NS and R-R sectors respectively, with field strengths $H_3 = dB_2$ and $F_3 = dC_2$. In addition this theory has a four form field C_4^+ whose field strength $F_5 = *F_5$, i.e. is self–dual. The Bianchi identity satisfied by this field is $dF_5 = H_3 \wedge F_3$ so that $F_5 = F_5' - H_3 \wedge C_2$, where locally we may write $F_5' = dC_4$.[16] Now if one imposes the self duality on the fields in the Lagrangian it is not possible to write a kinetic term for C_4. Instead one imposes the self duality constraint only at the level of the equations of motion (i.e. only the solutions must satisfy this constraint); but one then needs to add a topolgical term to the action in order that the equations of motion be consistent with the Bianchi identity. The terms in question are

$$\frac{1}{2\kappa^2} \int_{M_{10}} \left(\frac{1}{2} F_5 \wedge *F_5 + C_4 \wedge H_3 \wedge F_3 \right). \tag{3.25}$$

The equation of motion $d * F_5' = H_3 \wedge F_3$ is then the same as the Bianchi identity once the self-duality constraint is imposed. Now we have the magnetic charge equations,

$$\int_{S_3} H_3 = \frac{2\pi}{T_1}, \quad \int_{S_3} F_3 = \frac{2\pi}{T_1}, \quad \int_{S_5} F_5 = \frac{2\pi}{T_3}. \tag{3.26}$$

Then as in the previous argument for the type IIA case we have

$$\frac{1}{2\kappa^2} \int_{S_5 \times S_3 \times S_3} F_5 \wedge H_3 \wedge F_3 = \frac{1}{2\kappa^2} \frac{2\pi}{T_3} \left(\frac{2\pi}{T_1} \right)^2 = 2\pi, \tag{3.27}$$

where we again used (3.4), (3.5) to get the last equality. Note that on the left hand side of the first equality there is no factor two unlike in the IIA case. This is because now we have two different three forms in the integrand. Thus for instance if we choose y^1, y^2, y^3 as the coordinates on one S_3 and y^4, y^5, y^6 on the other, a minimal configuration is obtained by taking $H_3 = H_{123} dy^1 \wedge dy^2 \wedge dy^3$, $F_3 = F_{456} dy^4 \wedge dy^5 \wedge dy^6$ whereas in the previous (IIA) case a minimal configuration for the three form field strength is

[15] See [4] and references therein.

[16] See next section for a discussion of the gauge invariance of such terms.

$F_3 = F_{123} dy^1 \wedge dy^2 \wedge dy^3 + F_{456} dy^4 \wedge dy^5 \wedge dy^6$. The last equality in (3.27) comes from substituting for κ and the tensions from (3.5) and (3.4) and checks the consistency of the normalization.

4 Nonrenormalization of D-brane actions

The bosonic part of the D-p-brane action is given by,

$$S_p = -T_p \int_{W_{p+1}} d^3\xi e^{-\phi} \sqrt{\det(G_{ij} + \mathcal{F}_{ij})} - T_p \int_{W_{p+1}} C \wedge e^{\mathcal{F}_2} \tag{4.1}$$

In the above C stands for a formal sum of anti-symmetric tensors and $\mathcal{F}_2$ is given in the line after (3.18) and the appropriate sum of wedge products of C_r and $\mathcal{F}_2$ coming from the expansion of the exponential is understood . This form of the action can actually be derived in pertubation theory from the disc topology [16],[17], [18],[19], [13][14]. However we would like to argue that it is valid as a quantum exact but low energy and constant gauge field strength, effective action coming from integrating out sub-string scale fluctuations.

The action has a gauge invariance under, $C_r \to C_r + d\Lambda_{r-1} - H_3 \wedge \Lambda_{r-3}$. As is well known a topological term is not expected to get renormalized since for instance any ϕ dependence will spoil the gauge invariance of this term,[17] so one does not expect the second term in the action to be renormalized. As for the first term one needs to consider the following facts. As argued in section one the dilaton dependence when $\mathcal{F} = 0$ can be obtained by imposing some naturalness criteria and requiring the existence of solutions to the coupled equations of background plus brane. The question then is whether the Born-Infeld piece $\sqrt{\det(1 + G^{-1}\mathcal{F}}$ is valid (upto derivative terms in the field strength) without acquiring (dilaton dependent) renormalization.

The Born-Infeld (D-9-brane) action in ten dimensions was originally derived from (open) string theory by calculating the partition function on the disc by Fradkin and

[17]Even in field theory with a constant coupling such terms are not renormalized since the normalization is fixed by topological considerations.

Tseytlin [20]. Subsequently however it was shown by Aboulsaoud et al. [21] that the result can be obtained from the requirement of conformal invariance i.e. from a beta function calculation. Consequently, (as observed in that paper), one can argue that the result is independent of world sheet topology i.e. it should be valid to all orders in perturbation theory. The beta function argument was extended to $p < 9$ branes by Leigh [17]. The upshot is that at least to all orders in perturbation theory the form of the DBI action should be valid for all D-branes.

What about non-perturbative renormalization? For the D-2-brane one might think that one can go further since the D-brane action can be obtained from the M-theory 2-brane action. The latter is the strong coupling version of the former so that this fact might be interpreted as evidence for the non-renormalization (even non-perturbatively) of the former. However the equivalence is shown only at the level of treating some of the fields classically (see footnote 10) so that it is not clear that this establishes the needed result. However recent work has given some support to the conjecture that the form of the D-brane action is unrenormalized even non-perturbatively. This is the demonstration [23],[24] that the D-brane action must have a κ-symmetry which ensures that it has the right number of degrees of freedom consistent with supersymmetry. Given the above argument for the non-renormalization of the toplological term, the kappa symmetry seems to ensure that the Born-Infeld factor is also unrenormalized.

When N D-branes come together one expects the $U(1)^N$ symmetry to be enhanced to $U(N)$ [25]. Since in the limit when the gauge field configuration is in the Cartan sub algebra we should get for the sum of the diagonal terms each of which is the abelian action the D-brane action should get replaced by

$$S_p = -T_p \int_{W_{p+1}} d^3\xi e^{-\phi} S\mathrm{tr}\sqrt{\det(G_{ij} + \mathcal{F}_{2ij})} - T_p \int_{W_{p+1}} C \wedge \mathrm{tr} e^{\mathcal{F}_2} + \dots \tag{4.2}$$

where the error would be terms which vanish in the abelian limit. i.e. commutator terms (including terms for the dynamical transverse position variables of the D-brane) . (See [26] for a detailed discussion). Thus up to such terms the above should be a valid quantum effective action for N concident D-branes. In the next section we will use these

results (for $p = 9$) to argue that gauge coupling terms in type one and heterotic theories are fixed non-perturbatively at their lowest order values.

5 Type I strings, heterotic strings and M-theory

So far our discussion has been limited to type IIA, M-theory on smooth manifolds, and type IIB string theory. Let us now extend it to type I and heterotic strings. The discussion is a modified and extended version of that given in [5].

Type I strings are obtained from type IIB by making an orientation projection and adding (for consistency) SO(32) open strings D-ninebranes (see [3] and references therein). To proceed we need to assume that the relation between the string scale $\sqrt{\alpha'}$ and κ is the same as in type II. Since the projection leading to type I from type II does not affect the gravitational interactions we believe this is a very plausible assumption. The low energy effective action for this theory may be written,

$$\begin{aligned} I_I &= -\frac{1}{(2\pi)^7\alpha'^4}\int_{M_{10}}[e^{-2\phi}\sqrt{-G}(R-4(\nabla\phi)^2)-\frac{1}{2}F_3\wedge *F_3] \\ &\quad -\frac{1}{(2\pi)^9\alpha'^5}\int_{M_{10}}e^{-\phi}\mathrm{tr}(\sqrt{\det(G_{\mu\nu}-2\pi\alpha' F_{\mu\nu})}-\sqrt{-G}) \\ &\quad -\frac{(2\pi\alpha')^4}{(2\pi)^9\alpha'^5}\int_{M_{10}}C_2\wedge\mathrm{tr}F_2^4+\ldots \end{aligned} \tag{5.1}$$

In the last two lines we have the 9-D-brane action with its tension given by (3.4).

The gauge field terms come from the D-9-brane and are not expected to get renormalized as argued in section 4. Note that the leading (two derivative) gauge field term is ,

$$-\frac{1}{4(2\pi)^7\alpha'^3}\int_{M_{10}}\sqrt{-G}\mathrm{tr}F^2 \tag{5.2}$$

By the field redefinitions, $G_{\mu\nu} = e^{-\phi_H}G_{H\mu\nu}$ and $e^{\phi} = e^{-\phi_H}$ we get the low energy effective action of the SO(32) heterotic string theory:

$$
\begin{aligned}
I_H \;=\; & -\frac{1}{(2\pi)^7\alpha'^4}\int_{M_{10}}\sqrt{-G}e^{-2\phi_H}\left\{R-4(\nabla\phi)^2+\frac{1}{12}F_3^2\right\} \\
& -\frac{1}{(2\pi)^9\alpha'^5}\int_{M_{10}} e^{\phi_H}\mathrm{tr}(\sqrt{\det(e^{-\phi_H}G_{H\mu\nu}-2\pi\alpha' F_{\mu\nu})}-e^{-5\phi_H}\sqrt{-G_H}) \\
& -\frac{(2\pi\alpha')^4}{(2\pi)^9\alpha'^5}\int_{M_{10}} C_2\wedge \mathrm{tr}F_2^4 \qquad (5.3)
\end{aligned}
$$

(In the above the R and the contractions are defined with respect to G_H). Now the leading gauge field term is

$$
-\frac{1}{4(2\pi)^7\alpha'^3}\int_{M_{10}}\sqrt{-G_H}e^{-2\phi_H}\mathrm{tr}F^2 \qquad (5.4)
$$

Note that the ratio of the gauge field coupling to the gravitational coupling is exactly as predicted to all orders in perturbation theory. It is instructive to expand the square root determinant terms in these two actions to $O(F^4)$ and then compare the results with standard string perturbation theory calculations. To the extent that the terms have been calculated they are a check on S-duality [27].

Now the $E_8\times E_8$ heterotic theory is related to the $SO(32)$ one by T-duality. Hence the leading terms in this action will be

$$
\begin{aligned}
I_{H'} \;=\; & -\frac{1}{(2\pi)^7\alpha'^4}\int_{M_{10}}\sqrt{-G_{H'}}e^{-2\phi_{H'}}\left\{R-4(\nabla\phi)^2+\frac{1}{12}F_3^2\right\} \\
& -\sum_i\frac{1}{4(2\pi)^7\alpha'^3}\int_{M_{10}}\sqrt{-G_{H'}}e^{-2\phi_{H'}}\mathrm{tr}F_i^2 \qquad (5.5)
\end{aligned}
$$

Note that as far as the gravitational terms go this is the same transformation as that taking us from type IIB to type IIA. One takes a configuration with an isometry and compactifies in that direction on a circle of radius R. The T-dual theory is then on a circle of radius $R'=\frac{\alpha'}{R}$ and the dilaton is related by $e^{\phi_{H'}}=\frac{\sqrt{\alpha'}}{R}e^{\phi_H}$. In the present case however one has also to introduce a Wilson line which breaks the $SO(32)$ symmetry to $SO(16)\times SO(16)$. Upon letting $R'\to\infty$ one gets a ten-dimensional theory with $E_8\times E_8$ symmetry[18] [28],[29]. The gauge to gravitational coupling ratio is fixed to all orders in heterotic perturbation theory by a well-known argument [30].

[18] We note that the further transformation $G_{H'}=e^{\phi_{H'}}G'_{\mu\nu},\ e^{\phi'}=e^{\phi_{H'}}$ gives us a form of the action

The strong coupling version of this theory has been identified [31] as M theory on an orbifold, $M_{10}\times\frac{S_1}{Z_2}$ by Horava and Witten. These authors determined (by requiring gauge anomaly cancellation) the dimensionless ratio of gauge to gravitational constants,(see also [32]). With our value for κ_{11} the action in question (in upstairs form - i.e. on the smooth manifold $M_{10}\times S_1$ with a Z_2 symmetry on the fields) is (omitting the topological terms)

$$\begin{aligned} S_{H'} &= -\frac{1}{(2\pi)^8\alpha'^{\frac{9}{2}}}\int_M\sqrt{-G_M}\left\{R+\frac{1}{48}K_4^2\right\} \\ &\quad -\sum_i\frac{1}{4(2\pi)^7\alpha'^3}\int_{M_{10}^i}\sqrt{-G_{H'}}e^{-2\phi_{H'}}\mathrm{tr}F_i^2 \end{aligned} \tag{5.7}$$

In the above $M_{10}^i,\ i=1,2$ are the ten dimensional manifolds at the Z_2 fixed points on which the gauge fields live.

Let us now check the calculation of the gauge coupling constant by comparing to the standard form of the heterotic effective action. The metric transformation that needs to be done is almost the same as that which takes us from the M-theory to IIA, namely,

$$ds_M^2=e^{-\frac{2}{3}\phi_{H'}(x)}G_{H'\mu\nu}(x)dx^\mu dx^\nu+e^{\frac{4}{3}\phi_{H'}(x)}dy^2 \tag{5.8}$$

(On the fixed ten dimensional planes the RR fields C_1,C_3 restricted to these planes disappear because of the symmetry (under for instance $y(=x^{11})\to -y)$). Then (under the assumption that the fields are independent of the eleventh coordinate) the action (5.7) goes over into (5.5). Note that the fact that the gauge coupling that was fixed at

that is similar to type I and maybe called type $\overline{I}$.

$$\begin{aligned} I_{\overline{I}} &= -\frac{1}{(2\pi)^7\alpha'^4}\int_{M_{10}}[e^{-2\phi}\sqrt{-G'}(R-4(\nabla\phi)^2)-\frac{1}{2}F_3\wedge *F_3] \\ &\quad -\sum_i\frac{1}{4(2\pi)^7\alpha'^3}\int_{M_{10}^i}\sqrt{-G'}e^{-\phi'}\mathrm{tr}F_i^2 \end{aligned} \tag{5.6}$$

One might have expected that the above is the effective low energy action coming from the type IA (or I') theory discussed in [33]. However this does not appear to be the case. The type IA theory with coupling constant $e^{\phi_{IA}}$ compactified on a circle of radius R_{IA} is related to the E_8 theory with coupling constant $e^{\phi_{H'}}$ on a circle of radius R_8 by the relations $R_{IA}^2=e^{\phi_{H'}}R_8,\ e^{\frac{2}{3}\phi_{IA}}=R_8e^{-\frac{1}{3}\phi_{H'}}$ which is not in agreement with the above relations to what we have called $\overline{I}$.

the M-theory level by anomaly cancellation, goes over exactly to the gauge coupling as fixed in string theory, is an independent check on the Horava-Witten theory.

6 2-brane and 5-brane couplings in M-theory

In this section we discuss the couplings and normalizations of the membrane and the five-brane of M-theory. In particular we discuss how to obtain an action which incorporates five branes and membranes (with the latter having possible boundaries sitting on five-branes [9][8]), coupled to 11-D supergravity. This demonstrates the consistency of the picture developed for these interactions and also suggests a possible resolution to a problem with five-brane anomalies pointed out recently [7]. The discussion will in effect extend earlier work [8],[34],[35],[36],[7], on these matters.

The low energy effective action of M-theory is given in its standard form by equation (3.6). It is to this form of the supergravity action that the membrane couples naturally. There should however be another dual form of the action in terms of a seven form field strength $K_7 = *K_4 = dC_6 + \ldots$ to which the five-brane which is the magnetic dual of the membrane naturally couples. Because of the existence of the gauge field dependent topological term in (3.6) the standard dualization procedure cannot be carried out, and the complete low energy action for M-theory coupling to five branes has not been given.[19] We hope in this section to remedy this by giving the bosonic terms in this action. In fact we will present one action from which all the relevent equations can be obtained.

It is convenient to begin with the brane actions. For a closed membrane there is in addtion to the volume term $\int_{W_2} \sqrt{-detG_{ij}}$ (where G_{ij} is the pull-back of the M-theory metric to the world volume W_2 of the 2-brane) the coupling $\int_{W_3} C_3$ to the three form field[20]

[19] Another reason is the existence of a self-dual field strength on the five-brane.

[20] Here and in all subsequent formulae we will not explicitly write the pull-back map which is necessarily there in order to write the integral over the world volume of a field which is defined on the ambient M-theory space.

When the membrane has a boundary however this has to be modified since the above term is no longer gauge invariant under the standard M-theory gauge transformation $C_3 \to C_3 + d\Lambda_2$. One needs then an additional field on the boundary whose gauge transormation will cancel the boundary contribution from the above piece. Specifically we need

$$\int_{W_3} C_3 - \int_{\partial W_3} b_2 \tag{6.1}$$

where b_2 is a two form field living on the two dimensional boundary of the membrane world volume with the gauge transformation $b_2 \to b_2 + \Lambda_2$. Where can this field come from? The only possibility is that the boundary sits on a five-brane world volume [8],[9]. The latter is known to have a self-dual gauge field strength h_3. However we cannot just have $h_3 = db_2$ since this would not be invariant under the above gauge transformation. The correct modification is clearly [8], [4],

$$h_3 = db_2 - C_3|_{W_6} \tag{6.2}$$

Note that this implies the Bianchi identity

$$dh_3 = -dC_3|_{W_6} = -K_4|_{W_6} \tag{6.3}$$

This also means, since h_3 must be globally defined on W_6, that $T_2 K_4/2\pi$ (which in general has integral or half integral periods) actually has vanishing periods when restricted to the world volume of the five-brane [7].

What then is the action for this field? The problem is that it is self-dual. We will adopt here the strategy used in earlier work (see [4] and references therein), namely to impose the self-duality constraint at the level of the equations of motion and then require consistency with the Bianchi identity. Including also the coupling to the six form field C_6 this turns out to be

$$-\frac{1}{2}\int_{W_6} h_3 \wedge *h_3 - 2\int_{W_6} C_6 + \int_{W_6} C_3 \wedge db_2. \tag{6.4}$$

We will show below that self duality of h_3 and gauge invariance determine the (relative) coefficients as written above[21].

The action is to be used to derive the field equations and the self duality constraint $h_3 = *h_3$ is imposed at the level of the equations of motion. We get, by varying with respect to b_2,

$$d * h_3 = -dC_3 \tag{6.5}$$

Using the self duality condition this is exactly the Bianchi identity. Thus the relative coefficient of the first and last terms is fixed [4]. Now consider the Bianchi identity for the seven form field strength[22], K_7 (which is dual to K_4), which follows from the equation of motion for C_3 (see (3.6).

$$dK_7 = d * K_4 = -\frac{1}{2} K_4 \wedge K_4 \tag{6.6}$$

The solution of this equation is

$$K_7 = dC_6 - \frac{1}{2} C_3 \wedge K_4 \tag{6.7}$$

The invariance of the field strength K_7 under $\delta C_3 = d\Lambda_2$ then gives $\delta C_6 = \frac{1}{2}\Lambda_2 \wedge K_4$. It is then easily checked that the gauge invariant combination of topological terms is the one given in equation (6.4)[23]. Restoring the five-brane tension we then have the complete bosonic five-brane action

$$S_5 = T_5 \int_{W_6} \sqrt{-detG_{ij}} - \frac{T_5}{4} \int_{W_6} h_3 \wedge *h_3 - T_5 \int_{W_6} C_6 + \frac{T_5}{2} \int_{W_6} C_3 \wedge db_2. \tag{6.8}$$

Now this five-brane cannot couple to the original form of the action where the only independent variable (apart from gravity) is the three form field. This would leave C_6 without a kinetic term and its equation of motion will imply that $T_5 = 0$. We need the

[21] For recent discussion of the 5-brane action see [37], [38], [39].

[22] We shall ignore for the moment an additional piece coming from gravitational anomalies on the five-brane world volume [22]

[23] This seems to have been first pointed out in [35]

dual formulation of the 11 dimensional action in which $K_7 = dC_6 + \ldots$ appears. The difficulty with writing this down is the presence of the topological term which involves C_3 explicitly and so to our knowledge such an action has not been written down before. The following is perhaps a step toward solving this problem. Treating C_3, C_6 as independent fields we write down the (bosonic) action, (omitting the pure gravity piece).

$$\tilde{I}_{11} = -\frac{1}{2}\int K_7 \wedge *K_7 + \frac{1}{3}\int C_3 \wedge dC_3 \wedge dC_3 \tag{6.9}$$

The integrals in the above are taken over M as in (3.6).[24] Note that the coefficient of the topological term is twice that of (3.6). K_7 in the above action is defined by equation (6.7) with $K_4 = dC_3$. Now this action involves both C_3 and C_6 so it is not what one would normally call a dual action. In fact it is probably impossible to construct an action involving C_6 alone since the Bianchi identity for K_7 involves C_3. Nevertheless it is on a similar footing to the action for IIB supergravity[4] (which involves a self dual five form field strength) or the action (6.8) which involves a self-dual three form. As in those cases we need to impose a constraint, at the level of the field equations, that is consistent with them. In the present case this constraint is $*K_7 = -dC_3$. The variation of the action with respect to C_6 gives the equation of motion $d * K_7 = 0$. Variation with respect to C_3 gives, after using the previous equation, $dC_3 \wedge (*K_7 + dC_3) = 0$. This is clearly consistent with our constraint. [25]

The above action enables us to consider C_6 as a dynamical field and now we may couple two and five branes. (It is not obvious how to do this with the original form of the M-theory action (3.6)). Thus we have the action $I = \tilde{I}_{11} + S_2 + S_5$ where the first term on the right hand side is given by (6.9), the second by $-T_2 \int_{W_3} C_3 + T_2 \int_{\partial W_3} b_2$, and the third by (6.8). The independent fields are C_3, C_6, and b_2.

It turns out that the definition of h_3 needs to be modified because of the non-zero boundary of W_3 i.e. we now must have $h_3 = db_2 - C_3|_{W_6} + \frac{2T_2}{T_5}\theta_3(W_6 \to W_3)$ where θ_3 the

[24] In the rest of this section an integral whose range is not indicated is to be taken over M.

[25] I wish to thank Eugen Cremmer, Bernard Julia, Hong Lu, and Chris Pope for pointing out an incorrect statement in a previous version of this argument.

restriction of M to W_3 is defined below.

In order to derive the equations of motion in the presence of these sources it is convenient to introduce delta (step) function differential forms which enable us to write all the terms as integrals over M. Thus we put for the integral of an arbitrary 6-form A_6, (for which we assume that an extension to M with compact support exists),

$$\int_{W_6} A_6 = \int_M A_6 \wedge \delta_5 \tag{6.10}$$

Here δ_5 is a closed delta function 5-form which is a natural genaralization of the delta one-form $\delta(x)dx$ (see (2.6), (2.7)). Note that this makes sense only if W_6 is closed[26] (i.e. $\partial W_6 = \emptyset$. By contrast consider the seven manifold W_7^+ which has a non-zero boundary W_6. Now the restriction must be given by a generalization of a stepfunction times a delta function form θ_4 which has compact support in the directions normal to W_7^+ . Let us integrate a seven form $F_7 = dA_6$ over W_7^+

$$\int_{W_7^+} dA_6 = \int_M dA_6 \wedge \theta_4 = -\int_M A_6 \wedge d\theta_4. \tag{6.11}$$

But from Stokes' theorem the left hand side is equal to the left hand side of (6.10). Hence consistency requires $d\theta_4 = -\delta_5$. We may deal with the integral over W_3 (whose boundary is non-null) in a similar fashion i.e. by writing $\int_{W_3} C_3 = \int_M C_3 \wedge \theta_8$.

Now the equations of motion for the combined action may be derived and we get,

$$\begin{aligned}
\delta C_6: \quad d * K_7 &= T_5 \delta_5 (M \to W_6) \\
\delta C_3: \quad -dC_3 \wedge *K_7 &= dC_3 \wedge dC_3 - 2T_2 \theta_8 (M \to W_3) + T_5 h_3 \wedge \delta_5 \\
\delta b_2: \quad dh_3 &= -dC_3|_{W_6} - \frac{2T_2}{T_5} \delta_4 (W_6 \to \partial W_3).
\end{aligned} \tag{6.12}$$

In the above we have again used the self-duality constraint for h_3. The constraint relating C_6 and C_3 has to be changed now to be consistent with the second equation. Thus we need to put $*K_7 = -dC_3 - T_5 \theta_4$ where $d\theta_4 = -\delta_5(M \to W_6)$. Similarly the modification of the definintion of h_3 mentioned earlier is necessary in order to have compatibility with

[26] Consider $A_6 = d\Lambda_5$. Then since δ is closed the right hand side vanishes for arbitrary Λ_5 vanishing at infinity in M, but the left hand side vanishes only if W_6 is closed.

the last equation. Note that taking the exterior derivative of the second equation and using the first, yields an equation that is consistent with the third.

Let us now write down a self-dual form of the M-theory action.

$$I_{sd} = -\frac{1}{2}\int K_4 \wedge *K_4 - \frac{1}{6}\int C_3 \wedge dC_3 \wedge dC_3 + \int K_7 \wedge (K_4 - dC_3). \tag{6.13}$$

Here K_4, K_7 and C_3 are to be treated as independent fields. Thus we have the following field equations.

$$\begin{aligned} \delta_{K_7}: \quad K_4 &= dC_3, \\ \delta_{K_4}: \quad *K_4 &= K_7, \\ \delta_{C_3}: \quad dK_7 &= -\frac{1}{2} dC_3 \wedge dC_3. \end{aligned} \tag{6.14}$$

If the first equation is substituted into the action (6.13) we get the normal form of the action i.e. equation (3.6). If on the other hand in (6.13) we use the second equation (and its dual $K_4 = - * K_7$) as well as the solution $K_7 = dC_6 - \frac{1}{2} C_3 \wedge dC_3$ to the constraint given in the last equation, we get the dual form (6.9).

Now consider the coupling of 2- and 5-branes to this action. The action becomes,

$$\begin{aligned} I_{sd} + S_2 + S_5 = & -\frac{1}{2}\int_M K_4 \wedge *K_4 - \frac{1}{6}\int_M C_3 \wedge dC_3 \wedge dC_3 + \int_M K_7 \wedge (K_4 - dC_3) \\ & -T_2 \int_{W_3} C_3 + T_2 \int_{\partial W_3} b_2 - \frac{T_5}{4}\int_{W_6} h_3 \wedge *h_3 \\ & -T_5 \int_{W_7^+} (K_7 + \frac{1}{2} K_4 \wedge C_3) + \frac{T_5}{2}\int_{W_7^+} K_4 \wedge db_2 \end{aligned} \tag{6.15}$$

In the above we take M and W_6 to be closed manifolds but $\partial W_3 \neq 0$. The manifold W_7^+ is such that its boundary $\partial W_7^+ = W_6$. Note that the integrals over W_7^+ are in fact a rewriting of the topological integrals over W_6 in (6.8). In the above action however the independent fields are taken to be C_3, K_4, K_7 and b_2. Note that this form of the action (i.e. involving K_7 rather than C_6 (and hence also an integral over the open disc W_7 rather than the closed manifold W_6) is forced upon us by the requirement of a consistent coupling to

the M-theory background. As before we define $h_3 = db_2 - C_3 + (2T_2/T_5)\theta_3$. The action is invariant under the gauge transformations $C_3 \rightarrow C_3 + d\Lambda_2,\ b_2 \rightarrow b_2 + \Lambda_2, h_3 \rightarrow h_3, K_4 \rightarrow K_4, K_7 \rightarrow K_7$.

Now the field equations are,

$$\begin{aligned}
\delta_{K_7}: \quad K_4 &= dC_3 + T_5\theta_4 \\
\delta_{K_4}: \quad *K_4 &= K_7 + \frac{T_5}{2} h_3 \wedge \theta_4 \\
\delta_{C_3}: \quad dK_7 &= -\frac{1}{2} dC_3 \wedge dC_3 - T_2\theta_8 + \frac{T_5}{2}(h_3 \wedge \delta_5 - K_4 \wedge \theta_4) \\
\delta_{b_2}: \quad dh_3 &= -K_4|_{W_6} - \frac{2T_2}{T_5}\delta_4(W_6 \rightarrow \partial W_3)
\end{aligned} \tag{6.16}$$

In the last two equations we have used the self-duality constraint $h_3 = *h_3$. Note that the consistency condition for the third equation, $d^2K_7 = 0$, follows from the fourth equation and the use of $\theta_4(M \rightarrow W_7^+) \wedge \delta_5(M \rightarrow W_6) = 0$.

We have demonstrated above the consistency of the picture that has been diccussed in the literature on the coupling of two- and five-branes in M-theory. The self-dual form in fact allows one to reexpress these couplings in terms of the original K_4 form of the supergravity action. Thus by substituting the first of equation (6.16) into the self-dual action (6.15) we have

$$\begin{aligned}
I = & -\frac{1}{2}\int_M K_4 \wedge *K_4 - \frac{1}{6}\int_M C_3 \wedge dC_3 \wedge dC_3 \\
& -T_2 \int_{W_3} C_3 + T_2 \int_{\partial W_3} b_2 - \frac{T_5}{4}\int_{W_6} h_3 \wedge *h_3 \\
& -\frac{T_5}{2}\int_{W_7^+} dC_3 \wedge h_3,
\end{aligned} \tag{6.17}$$

with $K_4 = dC_3 + T_5\theta_4$.

Note that even though in the dual form the couplings to the five-brane can be expressed in a purely six-dimensional form in the above we have to use explicitly the seven manifold W_7^+. This is the analog of the Dirac string. We will show in the next section that this and similar terms coming from perturbative anomalies can be written in a form that is manifestly independent of the seven manifold.[27]

[27] I wish to thank E. Witten for emphasizing the necessity of showing this.

7 Perturbative anomalies in the presence of five-branes

The self dual form of the M-theory low energy action and its coupling to two and five branes, enables us now to discuss the origin of a certain term that is necessary for the cancellation of (gravitational) anomalies in the Horava-Witten theory. This term is required because of the existence of gravitational (local Lorentz) anomalies on the 6D world volume of the five brane. The original discussion of this term has to be modified in the light of the observations in [7] so let us brefly review those.

Let TM be the tangent bundle on M. Its restriction to W_6 may be written as $TW \oplus N$ where the first term is the tangent bundle and the second the normal bundle on W_6. The anomalies in question are associated with local Lorentz tranformations preserving the direct sum structure of the bundle and come from two sources. They are the chiral fermion on the five-brane which couples to the spin connection on the brane world volume, and an $SO(5)$ gauge field associated with rotations of the normal bundle, and the chiral two form field b which 'sees' only tangent bundle rotations. As usual the anomaly is associated with index theory and characteristic classes on a manifold W_8 in two higher dimensions. As computed in [7] the chiral fermion anomaly is related by the usual descent formalism to the following eight form,

$$\frac{1}{2}I_{Dirac}(TW \oplus N) = \frac{1}{2}\left[4\frac{7(p_1(TW)/2)^2 - p_2(TW)}{1440} - \frac{p_1(N)p_1(TW)}{48} + \frac{p_1^2(N)}{96} + \frac{p_2(N)}{24}\right] \tag{7.1}$$

where $p_i(TW),\ p_i(N),\quad i = 1,2$, are the first and second Pontryagin classes of the tangent and normal bundles on W_6. The integral of I_{Dirac} is the Dirac index and the factor half is present because the fermion is (symplectic) Majorana-Weyl.

The chiral two form anomaly is related to,

$$I_A = \frac{1}{5760}[16p_1^2(TW) - 112p_2(TW) \tag{7.2}$$

Define $\Omega_8 = \frac{1}{48}(p_2 - (\lambda)^2)(TW \oplus N)$ (where $\lambda = p_1/2$). Then using $p_1(TM) =$

$p_1(TW) + p_1(N),\ p_2(TM) = p_2(TW) + p_1(TW)p_1(N) + p_2(N)$ one has the result that the total anomaly which needs to be cancelled is descended from,

$$\frac{1}{2}I_{Dirac} + I_A = -\Omega_8(TM) + \frac{p_2(N)}{24} \tag{7.3}$$

As observed in [7] the first term on the LHS can be cancelled by anomaly inflow from a term [40][22] proportional to $\int K_4 \wedge \Omega_7^{C.S.}$. [28] Here $\Omega_8 = d\Omega_7^{C.S.}$, with $\delta\Omega_7^{C.S.} = d\Omega_6^1$, so that using $dK_4 = -T_5\delta$ and Stokes' theorem we have the required anomaly cancelling variation. However as pointed out in that work the second term remains uncancelled.

A way to achieve anomaly cancellation in the type IIA case was given in [7]. Let us first give a (slightly modified) account of this. When $M = M_{10} \otimes S^1$ the normal bundle to the five-brane becomes $N = N' \oplus O$ where O is the trivial tangent bundle to S^1 so that $p_i(N) = p_i(N')$. The structure group of N' is $SO(4)$, so that $p_2(N') = \chi(N')^2$, where χ, the Euler character of N', is represented by $\chi(F) = F^{ab} \wedge F^{cd}\epsilon_{abcd}/32\pi^2$. The standard descent equations then read, $\chi(F) = d\chi_3^{C.S.},\ \delta\chi_3^{C.S.} = \chi_2^1$ where δ is a local $SO(5)$ variation. The anomalous variation that is to be cancelled is then proportional to $\chi_2^1 \wedge \chi(F)$, and hence can be cancelled by adding to the action the term

$$\frac{2\pi}{24}\int_{W_6} B|_W \wedge \chi(F), \tag{7.4}$$

where the two form field B of type IIA string theory, when restricted to W_6 must now have the variation $\delta B = -\chi_2^1$ under local gauge transformations. This however means (as in the usual Green-Schwarz mechanism) that the gauge invariant field strength is redefined to be $H|_{W_6} = dB|_{W_6} + \chi_3^{C.S.}$, which implies that $dH|_{W_6} = \chi(F)$. The equation implies that a necessary condition for this mechanism to work is that the Euler character χ, integrated over any four cycle on the five-brane world volume, must vanish. In this case $\chi_3^{C.S.}$ is globally defined and we may rewrite the counter term by using Stokes' theorem and the definition of $H|_{W_6}$ as

$$\frac{2\pi}{24}\int_{W_6} H|_W \wedge \chi_3^{C.S.} \tag{7.5}$$

[28] This term is usually written as $\int C_3 \wedge \Omega_8$ and in the absence of five branes it is equal to the term written in the text. In the presence of a five brane they are not the same because $K_4 \neq dC_3$.

which is the form given in [7]. On the 10d manifold in the presence of five branes, the Bianchi identity for H is modified to $dH = \delta$ (where δ is a delta function four form with support on W_6) and it was observed in [7] that when restricted to the five brane world volume, this reduces to the equation $dH|_{W_6} = \chi(F)$ given earlier, as a consequence of the theory of the Thom isomorphism. As pointed out in [7] it is not clear how to generalize this argument to M-theory. It was suggested there that one could use the formula

$$p_2(N) = \sum_e \chi_e \wedge \chi_e \tag{7.6}$$

where $\chi_e = F^{ab} \wedge F^{cd} \epsilon_{abcde}/32\pi^2$ with $a, \ldots e$ being $SO(5)$ indices in the normal bundle. It clearly reduces to the previous case when $N = N' \oplus O$. In M-theory unlike in type IIA there is no three form field strength so it was proposed that one should use (on W_6) the four form field K with one index in the normal bundle. i.e. introduce $K \sim H_a \wedge e^a$ where H_a is a three form field and e^a is a vielbein one-form which is (vector) valued in the normal bundle. It was suggested in [7] (in analogy with (7.5) that a counter term of the form $\int_{W_6}$ could be introduced to cancel the uncancelled anomaly in M-theory. Below we will try to work this suggestion out in some detail.

First we note that the structure group of the tangent bundle to M in the neighborhood of the five-brane is $SO(1,5) \otimes SO(5)$. Introducing a $SO(5)$ connection A^{ab} on the normal bundle to the five-brane, we have the covariant constancy equation for the vielbein one-form in the normal direction,

$$de^a + A^{ab} e^b = 0. \tag{7.7}$$

(Note that indices go over the vector representation of $SO(5)$ and are raised and lowered using the metric δ_{ab} on the fiber). The field strength $F^{ab} = dA^{ab} + A^{ac} \wedge A^{ca}$ is the curvature of the normal bundle and satisfies the Bianchi identity $DF^{ab} = dA^{ab} + (A \wedge F - F \wedge A)^{ab}$. Using this and the $SO(5)$ invariance of the epsilon symbol it follows that χ_a is covariantly constant. i.e.

$$(D\chi)_a = d\chi_a + A^{ab} \wedge \chi_b = 0 \tag{7.8}$$

(From this it is easily checked that $dp_2(N) = 0$ with p_2 given by (7.6), as should be the

case of course). Note that we have also have from the above the integrability conditions

$$F^{ab} \wedge e_b = F^{ab} \wedge \chi_b = 0. \tag{7.9}$$

Now we need to work out the relevant descent formalism. Consider the five form $\chi \equiv \chi_a \wedge e^a$. From the covariant constancy of the two factors it follows that $d\chi = 0$. Hence there is a Chern-Simons type four form, which may be written $\chi_a^{C.S.} \wedge e^a$ since it must scale linearly with e_a, such that

$$\chi_a \wedge e^a = d(\chi_a^{C.S.} \wedge e^a) = (d\chi_a^{C.S.} + A^{ab}\chi_b^{C.S.}) \wedge e^a \tag{7.10}$$

where we have used (7.7) and the the antisymmetry of the connection matrix. Thus we may set $\chi_a = d\chi_a^{C.S.} + A^{ab}\chi_b^{C.S.}$. Note that this works because of (7.9) and that it also implies $F^{ab} \wedge \chi_b^{C.S.} = 0$. Now since $\chi_a \wedge e^a$ is gauge invariant we have,

$$\delta(\chi_a^{C.S.} \wedge e^a) = d(\chi_a(\alpha, A) \wedge e^a) = (d\chi_a(\alpha, A) + A^{ab}\chi_b(\alpha, A)) \wedge e^a \tag{7.11}$$

where we have again used the covariant constancy of e^a. In the above α is a local $SO(5)$ gauge parameter so $\delta e^a = \alpha^{ab}e^b$, $\delta A^{ab} = [\alpha, A]^{ab} + d\alpha^{ab}$. Thus we may write, $\delta\chi_a^{C.S.} = d\chi_a(\alpha, A) + A^{ab}\chi_a(\alpha, A) + (\alpha\chi(\alpha, A))_a$. The anomaly coming from $p_2(N)$ via this modified descent formalism is

$$\sum_a \chi_a(\alpha, A) \wedge \chi_a(F). \tag{7.12}$$

Now the component of K on W_6 with one index in the normal bundle may be witten as $H_a \wedge e^a$. Since this must be closed we have locally (using again the covariant constancy of e) $H_a = dB_a + A_{ab} \wedge B_b$ where $B_a \wedge e^a \sim C$ is the component of the M-theory three form field C with one index in the normal bundle. The transformation $\delta C = d\Lambda$ induces the transformation $\delta B_a = (D\Lambda)_a$ and hence $\delta H_a = F_{ab} \wedge \Lambda_b$ but $H_a \wedge e^a$ is of course invariant as a consequence of (7.9). We may then write an anomaly cancelling term on the world volume of the five-brane as,

$$L = \int_{W_6} B_a \wedge \chi_a(F). \tag{7.13}$$

It should be observed that this is invariant under the Λ gauge transformation of B_a as a result of Stokes' theorem and the covariant constancy of $\chi_a(F)$. In order to cancel the anomaly we require, under $SO(5)$ gauge transformation, the variation

$$\delta B_a = -\chi_a(\alpha, A) + (\alpha B)_a \tag{7.14}$$

Now since the field strength $H_a \wedge e^a$ needs to be invariant we must replace its relation to B by,

$$H_a = (d + A)_{ab} B_b + \chi_a^{C.S.}(A) \tag{7.15}$$

so that it transforms covariantly under α gauge transformation. Using the above descent formalism we get the Bianchi identity for this field,

$$(DH)_a = F_{ab} \wedge B_b + \chi_a(F). \tag{7.16}$$

From this we have, using the covariant constancy of e^a and the equation (7.9)

$$d(H_a \wedge e^a) = (DH)_a \wedge e^a = \chi_a(F) \wedge e^e. \tag{7.17}$$

Integrating over any five cycle in W_6 we have the condition,

$$\int F^{ab} \wedge F^{cd} \wedge e^e \epsilon_{abcde} = 0. \tag{7.18}$$

It appears then that we can cancel the anomaly coming from $p_2(N)$ provided that we require five-branes to satisfy the above restriction. However there remains the problem of how to reconcile the H_a as defined above on W_6 with the four form field defined on M. In type IIA this was achieved as a result of the Thom isomorphism (see discussion after (7.5). In the present case the Bianchi identity for K in the presence of five-branes is $dK \sim \delta(M \to W_6)$. Picking up the relevant part of this equation we have (writing also $\delta(M \to W_6) \sim \delta_a \wedge e^a$) the equation $d(H_a \wedge e^a)|_W = (\delta_a \wedge e^a)|_{W_6}$. We then need the result that the 'finite part of' the delta function can be chosen to be equal to the left hand side of (7.17).

In the above approach the anomaly normal bundle anomaly discovered in [7] appears to be treated differently from the other anomalies which are cancelled by anomaly inflow

from the bulk. Therefore let us take a somewhat different approch which gives a unified treatment. This will also involve the assumption made at the end of the last paragraph and will turn out to be closely related to the previous discussion.

It is convenient to introduce the following redefinitions of our fields.

$$c_3 = \frac{T_2 C_3}{2\pi},\ c_6 = \frac{T_5 C_6}{2\pi},\ x_4 = dc_3,\ x_7 = dc_6 + \frac{1}{2} c_3 \wedge x_4,\ b = \frac{T_2 b_2}{2\pi},\ h = \frac{T_2 h_3}{2\pi}. \quad (7.19)$$

In the presence of five branes we have $dx_4 = \delta_5$. Since we have put $2\kappa_{11}^2 = 1$ we find $T_2 = (2\pi)^{2/3}$, $T_5 = (2\pi)^{1/3}$. Also from the Hodge decomposition theorem we may write, $x_4 = x_4' + \theta_4$ where the first term is a (unique) closed form (locally equal to dc_3) and the second is a (unique) coexact form. So $d\theta_4 = \delta_5$ and $d * \theta = 0$. It should also be noted that with these normalizations the closed forms x_4' and $x_7' = dc_6$ (locally) define integral classes. In particular $\int_{W_4} x_4 \epsilon \mathcal{Z}$, $\int_{W_7} x_7' \epsilon \mathcal{Z}$ (when $[\lambda]$ is even integral).

We start again with the dual form of the M-theory action coupled to 2- and 5-branes. It should be emphasized that we are using this action only as a guide to discovering the proper form of the five- and two-brane couplings in the usual form of the action and not as an end in itself. We have,

$$\begin{aligned} I &= -(2\pi)^{4/3} \frac{1}{2} \int x_7 \wedge * x_7 + \frac{2\pi}{3} \int c_3 \wedge x_4' \wedge x_4' + \\ & -2\pi \int_{W_3} c_3 + 2\pi \int_{\partial W_3} b - \frac{2\pi}{4} \int_{W_6} h \wedge * h - 2\pi \int_{W_6} c_6 + \frac{2\pi}{2} \int_{W_6} c_3 \wedge db + \ldots \end{aligned} \quad (7.20)$$

(The ellipses stand for pure gravity terms). Writing (using the standard descent formalism) $\Omega_8 = d\Omega_7^{C.S.}$, $\frac{p_2}{24} = d\omega_7$ and $\delta\Omega_7^{C.S.} = d\Omega_6^1$, $\delta\omega_7 = d\omega_6^1$, where δ is a variation in the local Lorentz group, $SO(5,1) \times SO(5)$ in the first case and $SO(5)$ in the second, the anomaly coming from (7.3) can be cancelled (as in the Green-Schwarz mechanism) by assigning a local Lorentz transformation to c_6 restricted to W_6, i.e.

$$\delta c_6 = -\Omega_6^1 (TW_6 \oplus N) + \omega_6^1(N). \quad (7.21)$$

Now again as in the Green-Schwarz mechanism the definition of the field strength x_7 needs to be changed from (7.19) since the latter is no longer gauge invariant. A neighbourhood of the five-brane world volume W_6 may be defined as a region R of M containing W_6 and

bounded by a manifold $W_6 \otimes S_4$. This is the region in which the structure group of the tangent bundle of M effevtively becomes $SO(5,1) \otimes SO(5)$ and its transverse size may be regarded as the thickness of the five-brane. Let δ_8 be a gauge invariant closed eight form with compact support in R such that $\delta_8|_R \sim \frac{p_2(N)}{24}$. Explicitly we may take (following the previous discussion) $\delta_8 = \sum_{a=1}^{5} \delta_{4a} \wedge \chi^a(F)$, where the second factor is defined after (7.6) and the first factor is a four form with compact support obtained by putting one index of the five-form $\delta(M \to W_6)$ in the normal bundle. Then what we need is (as in the discussion after (7.18) the generalization of the Thom isomorphism result used in [7], namely $\delta_{4a}|_{W_6} = \chi_a(F)$. Using the descent equations (see (7.12) and discussion) we have writing $\delta_8 = d\theta_7$ with $\theta_7 = \sum_a \delta_a \wedge \chi^{a\ C.S.}$ and $\delta\theta_7 = d\theta_6^1$ for a gauge variation, where $\theta_6^1 = \sum_a \delta_a \wedge \chi^a(A, \alpha)$ with $\theta_6^1|_{W_6} = \omega_6^1$. Using these we may extend (7.21) to M and define the gauge invariant field,

$$x_7 = dc_6 - \frac{1}{2} c_3 \wedge x_4' - \Omega_7^{C.S.}(TM) + \theta_7. \tag{7.22}$$

Taking the exterior derivative,

$$dx_7 = -\frac{1}{2} x_4' \wedge x_4' - \Omega_8 + \delta_8. \tag{7.23}$$

Outside R the last term is effectively zero and this relation becomes the usual one discussed in the literature [22]. Integrating over a closed[29] eight manifold we get a restriction namely,

$$I_8 = \int_{W_8} (-\frac{1}{2} x_4' \wedge x_4' - \Omega_8(TM) + \delta_8) = 0. \tag{7.24}$$

The dual formulation of supergravity is however somewhat problematic. It is not clear whether a supersymmetric version exists. Also the relation to the Horava-Witten theory

[29] Actually if we are in Minkowski space we ought to be talking about configurations with compact support. It is conceptually easier however to think in terms of compact spaces so that strictly speaking the rest of the discussion is valid as stated only provided it is understood that we have switched to a Euclidean metric on M. This also implies that the topological terms are to be understood as having factors of i.

is unclear. Nevertheless the above suggests a way to cancel the anomaly in the normal form of the theory. So let us see how this happens. Now there is no c_6 term so that it is not clear how to write down a local six dimensional term that cancels the anomalies. However it turns out that the relevant cancellation takes place by anomaly inflow from the eleven dimensional bulk. Thus we write the following expression for the complete (bosonic) action coupled to two and five branes:

$$\begin{aligned} I \;\; = \;\; & -\frac{1}{2}(2\pi)^{2/3}\int_M x_4\wedge *x_4+\frac{2\pi}{3}\int_M c_3\wedge x_4'\wedge x_4'-2\pi\int_M x_4\wedge(\frac{1}{2}x_4'\wedge c_3-\Omega_7+\theta_7) \\ & -2\pi\int_{W_3}c_3+2\pi\int_{\partial W_3}b_2-\frac{2\pi}{4}\int_{W_6}h\wedge *h-2\pi\int_{W_6}\frac{1}{2}x_4'\wedge b_2, \end{aligned} \tag{7.25}$$

with[30] $x_4 = x_4' + \theta_4$ and x_4' is the closed form which is locally equal to dc_3. Note that this is the version of the normal form of the M-theory action that is independent of any seven manifold that was promised at the end of the last section. It should be noted that it reduces (ignoring the quantum anomaly terms $\Omega,\ \theta$) to (6.17) when we choose $\theta_4 = -\theta_4(M \to W_7^+)$ where the left hand side is the step function times delta function restricting M to a particular open seven manifold, that was defined in the last section. It should also be noted that the above action is dual to the action (7.20).

The gauge variation of the third term (under the transformations $\delta c_3 = d\Lambda$,and local Lorentz transformations) gives, after using Stokes' theorem and the modified Bianchi identity $dx_4 = \delta(M \to W_6)$,

$$-\int_M x_4\wedge(\frac{1}{2}x_4'\wedge d\Lambda_2-d\Omega_6^1+d\theta_6^1)=\int_{W_6}(\frac{1}{2}x_4'\wedge\Lambda_2-\Omega_6^1+\omega_6^1). \tag{7.26}$$

The first term in the integrand cancels the anomaly coming from the last term of (7.25) and the second and third cancel the anomaly coming from (7.3).

Now we need to argue that this term is well-defined. Note that in the absence of five branes $x_4 = x_4',\ \theta_7 = 0$ and (7.25) reduces to the standard action of M-theory . In particular the topological term is (after using the formula $\frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$) exactly the one

[30] The topological terms over M in this action (except for the θ term) appear to be essentially equivalent to those in [7].

which was shown to be well-defined in [15]. In the presence of five-branes however we need to look at this issue again. In fact it does not seem possible to show this in general. We will need to impose a topological restriction, namely that $[\lambda]/2 \equiv [p_1]/4\epsilon\mathcal{Z}$. The terms in question are (up to a factor 2π)

$$+\frac{1}{3}\int_M c_3 \wedge x_4' \wedge x_4' - \int_M x_4 \wedge L_7 \tag{7.27}$$

where $L_7 \equiv \frac{1}{2}x_4' \wedge c_3 - \Omega_7 + \theta_7$. Under the above topological restriction it follows from the arguments of [15] that the first term is well-defined, i.e. the integral over the closed 12-manifold $\frac{1}{3}\int x_4' \wedge x_4' \wedge x_4' = 0$. The second term may be rewritten by using (7.22),

$$\int_M x_4 \wedge x_7 - \int_M x_4 \wedge dc_6 = \int_M x_4 \wedge x_7 + \int_{W_6} c_6 \tag{7.28}$$

where we have used Stokes' theorem and the Bianchi identity $dx_4 = \delta$ in the last step. The first term is clearly well-defined and the second term is well-defined, if there is a well defined coupling of the six form field to the five-brane which is the case when the Dirac quantization condition is satisfied. Thus a well-defined action in the normal form exists provided that there is a well-defined dual formualtion. Of course the definition of the five-brane should also include the restriction (7.18).

Finally let us check that our M-theory counter-term reduces to the one that is given by Witten [7] in the IIA case. In that case the counter term in question is $\int_{M_{10}} H_3 \wedge \theta_7$ where now $\theta_7 \sim \delta(M_{10} \to W_6)\Omega_\chi^{C.S.}(F)$ (so that $\delta_8 \sim \delta(M \to W_6)\chi(F)$) - with $\delta_8|_R = \chi(F)^2 \sim p_2(N')$ - and $dH_3 \sim \delta(M \to W_6)$. The counter term thus becomes, $\sim \int_{W_6} H|_{W_6} \wedge \Omega_\chi^{C.S.}$ which is expression (7.5), the term given in [7].

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